

ON SOME MODULI OF COMPLEXES ON K3 SURFACES

JASON LO

ABSTRACT. We consider moduli stacks of Bridgeland semistable objects that previously had only set-theoretic identifications with Uhlenbeck compactification spaces in [LQ]. On a K3 surface X , we give examples where such a moduli stack is isomorphic to a moduli stack of slope semistable locally free sheaves on the Fourier-Mukai partner \hat{X} . This yields a morphism from the stack of Bridgeland semistable objects to a projective scheme, which induces an injection on closed points. It also allows us to extend a theorem of Bruzzo-Maciocia on Hilbert schemes to a statement on moduli of complexes.

1. INTRODUCTION

Fourier-Mukai transforms have been used extensively to understand relations between various moduli spaces of sheaves. More recently, they have also been used to understand relations between moduli of complexes and moduli of sheaves. For example, Bernardara-Hein considered complexes on elliptic surfaces [BH], and Hein-Ploog had examples on elliptic K3 surfaces [HP]. Maciocia-Meachan considered rank-one Bridgeland stable objects on Abelian surfaces [MM], while Minamide-Yanagida-Yoshioka considered the relations between Bridgeland semistable complexes and Gieseker semistable sheaves on Abelian and K3 surfaces in [MY, MYY2].

In [LQ, Theorem 5.4], it was shown that, on a polarised smooth projective surface (X, ω) , if the rank r and degree $c_1\omega$ of the objects are coprime and the slope $c_1\omega/r$ of the objects is not equal to $\beta\omega$, then all the polynomial semistable objects are stable, and the moduli stack of polynomial stable objects is isomorphic to the moduli scheme of μ_ω -stable sheaves; here, $\beta \in \text{Num}(X)_\mathbb{Q}$ is another parameter in the definition of polynomial stability. And when μ_ω is equal to $\beta\omega$ (part (iii) of [LQ, Theorem 5.4]), it was only shown that there is a set-theoretic bijection between the closed points of the moduli of polynomial semistable objects, and the Uhlenbeck compactification space of μ_ω -stable locally free sheaves. In this case, the polynomial semistable objects coincide with the Bridgeland semistable objects of phase 1.

In this paper, we consider the latter case above, i.e. we consider the moduli of Bridgeland semistable objects of slope $\mu_\omega = \beta\omega$. If X is a K3 surface, we show that the moduli $\mathcal{M}^{\sigma ss}$ of these objects is isomorphic to a moduli of μ -semistable locally free sheaves on the Fourier-Mukai partner of X under an extra assumption (Theorem 3.1). This yields a morphism of stacks from $\mathcal{M}^{\sigma ss}$ to a projective scheme, which induces an injection on the closed points (Corollary 3.2). This way, we obtain an algebro-geometric alternative to the statement in part (iii) of [LQ, Theorem 5.4].

2010 *Mathematics Subject Classification.* Primary 14J60; Secondary: 14D20, 14D23, 14J28.

Key words and phrases. Bridgeland stability, moduli stack, K3 surface, Fourier-Mukai transform.

Theorem 3.1 also extends a theorem of Bruzzo-Maciocia's [BM, Theorem 0.1]: they constructed an isomorphism between the Hilbert scheme of points and a moduli space of Gieseker stable locally free sheaves on a reflexive K3 surface, which now extends to an isomorphism between a moduli stack of Bridgeland semistable objects and a moduli stack of μ -semistable locally free sheaves on any K3 surface. We explain this along with the details of other consequences of Theorem 3.1 in Section 4.

The main ingredients in our proof of Theorem 3.1 are Huybrechts' results on derived equivalent K3 surfaces [Huy].

1.1. Preliminaries. For any scheme X , we write $D(X)$ to denote the bounded derived category of coherent sheaves on X . For an object $E \in D(X)$, we write $H^i(E)$ to denote the degree- i cohomology of E .

If (X, ω) is a polarised smooth projective variety and $E \in D(X)$, we write $\mu_\omega(E)$ to denote $c_1(E)\omega/\text{rk}(E)$. Furthermore, if $\beta \in \text{Num}(X)_\mathbb{Q}$, then we write $\mathcal{A}(\beta, \omega)$ to denote the Abelian subcategory of $D(X)$ consisting of 2-term complexes $E \in D(X)$ such that:

- all the Harder-Narasimhan factors of $H^0(E)$ have slope $\mu_\omega > \beta\omega$;
- $H^{-1}(E)$ is torsion-free, and all its Harder-Narasimhan factors have slope $\mu_\omega \leq \beta\omega$;
- $H^i(E) = 0$ for all $i \neq -1, 0$.

The Abelian category $\mathcal{A}(\beta, \omega)$ is the heart of a bounded t-structure on $D(X)$ that is obtained from the standard t-structure from tilting.

The only moduli of polynomial semistable objects discussed in the rest of this paper is as in part (iii) of [LQ, Theorem 5.4]. In this case, polynomial stability coincides with a Bridgeland stability σ on $D(X)$, and the polynomial semistable objects are exactly the σ -semistable objects of phase 1 in $\mathcal{A}(\beta, \omega)$. Therefore, from now on, we will refer to these semistable objects only as Bridgeland semistable objects. Note that β, ω are part of the definition of the polynomial/Bridgeland stability here. And for fixed β, ω , the central charge on the heart $\mathcal{A}(\beta, \omega)$ that we use is

$$\begin{aligned} Z(E) &:= - \int_X e^{-(\beta+i\omega)} \cdot ch(E) \quad \text{for } E \in \mathcal{A}(\beta, \omega) \\ (1.1) \quad &= \text{rk}(E) \frac{\omega^2}{2} + i(c_1(E)\omega - \text{rk}(E)\beta\omega) + c(E) \end{aligned}$$

where $c(E) := -ch_2(E) + c_1(E)\beta - \text{rk}(E)\beta^2/2$. The phase of a nonzero object $E \in \mathcal{A}(\beta, \omega)$ is defined to be the real number $\phi(E) \in (0, 1]$ satisfying

$$Z(E) \in \mathbb{R}_{>0} \cdot e^{i\phi(E)}.$$

From (1.1), it is clear that an object $E \in \mathcal{A}(\beta, \omega)$ of positive rank and of slope $\mu_\omega(E) := c_1(E)\omega/\text{rk}(E) = \beta\omega$ has phase 1 (maximal phase). The reader may refer to [LQ, Section 2] for details of the definitions of these stability conditions.

All the schemes will be over $k = \mathbb{C}$.

1.2. Acknowledgments. The author would like to thank Arend Bayer, Dan Edidin, Zhenbo Qin and Ziyu Zhang for helpful discussions.

2. K3 SURFACES

Throughout this section, suppose X and \hat{X} are two non-isomorphic, derived equivalent K3 surfaces. By [Huy, Proposition 4.1], \hat{X} is isomorphic to a fine moduli space of μ_ω -stable locally free sheaves on X , with respect to some ample class $\omega \in \text{Num}(X)_\mathbb{Q}$. Suppose the μ_ω -stable sheaves parametrised by \hat{X} have slope $\mu_\omega = \beta\omega$ for some $\beta \in \text{Num}(X)_\mathbb{Q}$. If we let \mathcal{E} denote the universal family on $X \times \hat{X}$, then the Fourier-Mukai transform

$$\Psi : D(X) \xrightarrow{\sim} D(\hat{X})$$

with kernel \mathcal{E} induces an equivalence from the Abelian category $\mathcal{A}(\beta, \omega)$ to another Abelian category $\mathcal{A}(\hat{\beta}, \hat{\omega})$ for some $\hat{\beta}, \hat{\omega} \in \text{Num}(\hat{X})_\mathbb{Q}$ where $\hat{\omega}$ is ample [Huy, Proposition 5.2].

Let $\sigma, \hat{\sigma}$ be the Bridgeland stabilities on X, \hat{X} defined using β, ω and $\hat{\beta}, \hat{\omega}$, respectively.

Lemma 2.1. *The Fourier-Mukai transform Ψ takes a σ -semistable object in $\mathcal{A}(\beta, \omega)$ of slope $\mu_\omega(E) = \beta\omega$ to a $\hat{\sigma}$ -semistable object in $\mathcal{A}(\hat{\beta}, \hat{\omega})$ of slope $\mu_{\hat{\omega}} = \hat{\beta}\hat{\omega}$.*

Proof. Take any σ -semistable object $E \in \mathcal{A}(\beta, \omega)$ of slope $\mu_\omega(E) = \beta\omega$ (so $H^{-1}(E)$ is necessarily nonzero). From [Bay, Lemma 4.2(c)], we know $H^{-1}(E)$ is μ_ω -semistable of slope $\beta\omega$ and $H^0(E)$ is a 0-dimensional sheaf.

Consider the Jordan-Hölder filtration of $H^{-1}(E)$ with respect to μ_ω -stability:

$$(2.1) \quad 0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m = H^{-1}(E),$$

where each $G_i := F_i/F_{i-1}$ is μ_ω -stable with slope $\beta\omega$. And then, for each i , we have the short exact sequence of coherent sheaves

$$(2.2) \quad 0 \rightarrow G_i \rightarrow G_i^{**} \rightarrow T_i \rightarrow 0$$

where G_i^{**} is a μ_ω -stable locally free sheaf of slope $\beta\omega$, and T_i is a 0-dimensional sheaf T_i ; this gives an exact triangle

$$(2.3) \quad T_i \rightarrow G_i[1] \rightarrow G_i^{**}[1] \rightarrow T_i[1]$$

for each i .

By [Huy, Proposition 2.2], the minimal objects in $\mathcal{A}(\beta, \omega)$ are of the following forms:

- the skyscraper sheaf $k(x)$ at a closed point $x \in X$, or
- $F[1]$ where F is a μ_ω -stable locally free sheaf with $\mu_\omega(F) = \beta\omega$.

Therefore, using the canonical exact triangle

$$H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \rightarrow H^{-1}(E)[2]$$

in $D(X)$, together with the filtration (2.1) and the exact triangles (2.3), we can construct E as a series of extensions by minimal objects in $\mathcal{A}(\beta, \omega)$. Since Ψ induces an equivalence between $\mathcal{A}(\beta, \omega)$ and $\mathcal{A}(\hat{\beta}, \hat{\omega})$, it takes minimal objects in $\mathcal{A}(\beta, \omega)$ to minimal objects in $\mathcal{A}(\hat{\beta}, \hat{\omega})$. And any object in $\mathcal{A}(\hat{\beta}, \hat{\omega})$ built from a series of extensions by minimal objects is $\hat{\sigma}$ -semistable with slope $\mu_{\hat{\omega}} = \hat{\beta}\hat{\omega}$. This completes the proof of the lemma. \square

Note that, for any $y \in \hat{X}$, the Chern character of the fibre \mathcal{E}_y of the universal family \mathcal{E} is independent of y .

Proposition 2.2. *For any σ -semistable object $E \in \mathcal{A}(\beta, \omega)$ of Chern character $ch = (-r, -\delta, n)$, with $r > 0$ and satisfying:*

- (a) $\delta\omega/r = \beta\omega$, i.e. the μ_ω -slope of E is $\beta\omega$, and
- (b) *for any of the Jordan-Hölder factor G of $H^{-1}(E)$ with respect to μ_ω -stability, we have $ch(G^*) \neq ch(\mathcal{E}_y)$,*

the transform $\Psi(E)$ is a $\mu_{\hat{\omega}}$ -semistable locally free sheaf of slope $\mu_{\hat{\omega}} = \hat{\beta}\hat{\omega}$.

In Lemma 2.3 below, we give two numerical conditions where condition (b) in Proposition 2.2 is satisfied.

Proof. We use the same notation as in the proof of Lemma 2.1, so that the G_i denote the Jordan-Hölder factors of $H^{-1}(E)$. For each i , the sheaf G_i^* is reflexive, hence locally free, and so $ch_2(G_i^*) = ch_2(G_i^{**})$. By condition (b) in the hypothesis, none of the G_i^{**} is isomorphic to any \mathcal{E}_y . Since each G_i^{**} is a μ_ω -stable locally free sheaf with slope $\mu_\omega = \beta\omega$, by [Huy, Proposition 7.1], each of the transforms $\Psi(G_i^{**})$ is a $\mu_{\hat{\omega}}$ -stable locally free sheaf of slope $\hat{\beta}\hat{\omega}$ (up to shift). On the other hand, every 0-dimensional sheaf on X is mapped by Ψ to a $\mu_{\hat{\omega}}$ -stable locally free sheaf of slope $\hat{\beta}\hat{\omega}$. Therefore, using the exact triangle

$$H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \rightarrow H^{-1}(E)[2],$$

along with the filtration (2.1) and the triangles (2.3), we can construct $\Psi(E)$ by a series of extensions of $\mu_{\hat{\omega}}$ -stable locally free sheaves of slope $\hat{\beta}\hat{\omega}$, implying $\Psi(E)$ itself is a $\mu_{\hat{\omega}}$ -stable locally free sheaves of slope $\hat{\beta}\hat{\omega}$, sitting at degree -1 . \square

Lemma 2.3. *With the same notation as in Proposition 2.2, condition (b) of Proposition 2.2 is satisfied when:*

- (i) $0 < r < \text{rank}(\mathcal{E}_y)$, or
- (ii)

$$(2.4) \quad ch_2(\mathcal{E}_y) < - \left| n + (r-1) \frac{(\delta\omega)^2}{2\omega^2} \right|.$$

Proof. That the inequality in (i) implies condition (b) is clear. Now, suppose $ch(E) = (-r, -\delta, n)$ satisfies the inequality (2.4) in (ii). Using the same notation as in the proof of Proposition 2.2, we show that for each i , we have $ch_2(G_i^{**}) \neq ch_2(\mathcal{E}_y)$:

For any i , write $f_i := c_1(G_i)$ and $g_i := ch_2(G_i)$. Then for all i , we have $\mu_\omega(H^{-1}(E)) = \mu_\omega(G_i)$, i.e. $\delta\omega/r = f_i\omega/\text{rk}(G_i)$, i.e. $f_i\omega = \delta\omega \cdot \text{rk}(G_i)/r$. By [LQ, Lemma 3.7(i)], we have

$$f_i^2 \leq \frac{\left(\delta\omega \cdot \frac{\text{rk}(G_i)}{r} \right)^2}{\omega^2} \leq \frac{(\delta\omega)^2}{\omega^2}.$$

On the other hand, Bogomolov's inequality on each G_i gives us $g_i \leq f_i^2/(2\text{rk}(G_i))$. So overall, we have

$$(2.5) \quad g_i \leq \frac{f_i^2}{2\text{rk}(G_i)} \leq \frac{(\delta\omega)^2}{2\omega^2}.$$

Also, since $ch_2(H^0(E)) \geq 0$ and $n = -ch_2(H^{-1}(E)) + ch_2(H^0(E))$, we have

$$(2.6) \quad -n \leq ch_2(H^{-1}(E)) = \sum_{i=1}^m g_i.$$

The inequality (2.5) and equation (2.6) together give, for all i ,

$$(-|g_i|) + (m-1)\frac{(\delta\omega)^2}{2\omega^2} \geq -n,$$

i.e.

$$|g_i| \leq n + (m-1)\frac{(\delta\omega)^2}{2\omega^2}.$$

Since $m \leq r$, we have $|g_i| \leq n + (r-1)\frac{(\delta\omega)^2}{2\omega^2}$, hence

$$(2.7) \quad g_i \geq -\left|n + (r-1)\frac{(\delta\omega)^2}{2\omega^2}\right|$$

for all i . Therefore, by the inequality (2.4), $ch_2(G_i^*) = ch_2(G_i^{**}) \geq ch_2(G_i) = g_i > ch_2(\mathcal{E}_y)$ for any i and $y \in \hat{X}$. Hence $ch(G_i^{**}) \neq ch(\mathcal{E}_y)$ for any i . \square

Remark 2.4. In particular, if $r = 1$ (i.e. E is of rank 1), or $\delta\omega = 0$, then the inequality (2.4) reduces to $ch_2(\mathcal{E}_y) < -|n|$.

3. MODULI STACKS

Let X be any smooth projective K3 surface, and let σ be a Bridgeland stability as in Section 2.

Lieblisch [Lie] constructed an Artin stack \mathcal{M} of objects $E \in D(X)$ satisfying $\text{Ext}^{<0}(E, E) = 0$. We have various open substacks of \mathcal{M} :

- $\mathcal{M}_{X, ch}^{\sigma ss}$, the Artin stack of σ -semistable objects of Chern character ch on X , constructed by Toda [Tod];
- $\mathcal{M}_{X, ch}^{\mu ss}$, the Artin stack of μ_ω -semistable torsion-free sheaves of Chern character ch on X ;
- $\mathcal{M}_{X, ch}^{\mu ss, lf}$, the Artin stack of μ_ω -semistable locally free sheaves of Chern character ch on X , which is a substack of $\mathcal{M}_{X, ch}^{\mu ss}$.

We have:

Theorem 3.1. *Suppose all the objects in $\mathcal{M}_{X, ch}^{\sigma ss}$ satisfy the conditions of Proposition 2.2. Then the Fourier-Mukai transform $\Psi : D(X) \rightarrow D(\hat{X})$ induces an isomorphism of Artin stacks*

$$\Psi_{\mathcal{M}} : \mathcal{M}_{X, ch}^{\sigma ss} \rightarrow \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf}$$

(where \hat{ch} denotes the image of ch under Ψ^{ch}).

Let us set up some notation for the proof of the theorem. The equivalence $\Psi : D(X) \rightarrow D(\hat{X})$ has a quasi-inverse $\Phi : D(\hat{X}) \rightarrow D(X)$ that is also a Fourier-Mukai transform, with kernel $\mathcal{E}^*[1]$. For any scheme $S \rightarrow k$, we write X_S to denote $X \times_k S$. Then we have relative Fourier-Mukai transforms $\Psi_S : D(X_S) \rightarrow D(\hat{X}_S)$ and $\Phi_S : D(\hat{X}_S) \rightarrow D(X_S)$ induced by Ψ, Φ , respectively. For any closed point $s \in S$, we write X_s to denote the fibre of X_S over s , write j_s to denote the closed immersion $X_s \hookrightarrow X_S$ or $\hat{X}_s \hookrightarrow \hat{X}_S$, and write Ψ_s, Φ_s to denote the Fourier-Mukai transforms between the derived categories of X_s, \hat{X}_s induced by Ψ_S, Φ_S , respectively. See [BBR, Section 1.2.1] for more on relatively Fourier-Mukai transforms.

Proof. Take any scheme S over k . Let E_S be an S -flat family of σ -semistable objects on X of Chern character ch . By the base change formula [BBR, (1.15)]

$$(3.1) \quad Lj_s^* \Psi_S(E_S) \cong \Psi_s(Lj_s^* E_S)$$

and Proposition 2.2, we see that $\Psi_S(E_S)$ is an S -flat family of μ -semistable locally free sheaves on \hat{X} . This induces a morphism of stacks $\Psi_{\mathcal{M}} : \mathcal{M}_{X, ch}^{\sigma ss} \rightarrow \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf}$.

Conversely, suppose that F_S is an S -flat family of μ -semistable locally free sheaves of Chern character \hat{ch} on \hat{X} . Since each fibre of F_S over S is a $\hat{\sigma}$ -semistable object of slope $\mu_{\hat{\omega}} = \hat{\beta}\hat{\omega}$, by Lemma 2.1 and the base change formula above (both assertions are symmetric in X and \hat{X}), we see that Φ_S takes F_S to an S -flat family of σ -semistable objects of Chern character ch . This induces a morphism of stacks $\Phi_{\mathcal{M}} : \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf} \rightarrow \mathcal{M}_{X, ch}^{\sigma ss}$. Since Ψ and Φ are quasi-inverses to each other, so are $\Psi_{\mathcal{M}}$ and $\Phi_{\mathcal{M}}$. Hence $\Psi_{\mathcal{M}}$ is an isomorphism of stacks. \square

Given any smooth curve C on \hat{X} , let $\mathcal{M}_{C, (r, d)}^{ss}$ denote the moduli stack of μ -semistable (equivalently, Gieseker semistable) torsion-free sheaves on C of rank r and degree d . We know that $\mathcal{M}_{C, (r, d)}^{ss}$ admits a good moduli space

$$\gamma : \mathcal{M}_{C, (r, d)}^{ss} \rightarrow M_{C, (r, d)}^{ss}$$

in the sense of Alper, where $M_{C, (r, d)}^{ss}$ is a projective scheme [Alp, Example 8.7]. Recall that two closed points of $\mathcal{M}_{C, (r, d)}^{ss}$ are mapped to the same point of $M_{C, (r, d)}^{ss}$ if and only if they represent S -equivalent semistable sheaves on C .

Also, for any smooth curve C on \hat{X} , we can define a morphism of moduli stacks

$$\alpha : \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf} \rightarrow \mathcal{M}_{C, (r, d)}^{ss}$$

as follows: given any scheme S over k and any object $F_S \in \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf}(S) \subset D(\hat{X}_S)$, let α send F_S to the object $F_S|_{C \times S}$; here, we need $r = \hat{ch}_0$ and $d = \hat{ch}_1 \cdot [C]$.

Since the moduli stack $\mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf}$ is bounded, by [HL, Theorem 8.2.12], if the smooth curve C above further satisfies the requirement $C \in |m\hat{\omega}|$ for $m \gg 0$ depending on \hat{ch} , then two μ -semistable locally free sheaves F_1, F_2 of Chern character \hat{ch} on \hat{X} are isomorphic if and only if their restrictions $F_1|_C, F_2|_C$ are S -equivalent. That is, the composition

$$\gamma\alpha : \mathcal{M}_{\hat{X}, \hat{ch}}^{\mu ss, lf} \rightarrow \mathcal{M}_{C, (r, d)}^{ss} \rightarrow M_{C, (r, d)}^{ss}$$

induces an injection on the sets of closed points. Pre-composing $\gamma\alpha$ with the morphism $\Psi_{\mathcal{M}}$, we obtain:

Corollary 3.2. *When the objects in $\mathcal{M}_{X, ch}^{\sigma ss}$ satisfy the conditions of Proposition 2.2, and $C \in |m\hat{\omega}|$ is a smooth curve with $m \gg 0$ depending on \hat{ch} , we have a morphism of stacks*

$$\gamma\alpha\Psi_{\mathcal{M}} : \mathcal{M}_{X, ch}^{\sigma ss} \rightarrow M_{C, (r, d)}^{ss}$$

from the Artin stack of Bridgeland semistable objects $\mathcal{M}_{X, ch}^{\sigma ss}$ to the projective scheme $M_{C, (r, d)}^{ss}$ of semistable torsion-free sheaves on C , that induces an injection on the sets of closed points.

4. CONSEQUENCES AND APPLICATIONS

In part (iii) of [LQ, Theorem 5.4], it was shown that there is a set-theoretic bijection between the closed points of the moduli stack $\mathcal{M}_{X, ch}^{\sigma ss}$ from Section 3, and the closed points of the Uhlenbeck compactification space of μ -stable locally free sheaves on X ; this required the Bridgeland semistable objects to have rank and degree that are coprime. Corollary 3.2 gives us a morphism from the stack $\mathcal{M}_{X, ch}^{\sigma ss}$ to the scheme $M_{C, (r, d)}^{ss}$, which induces an injection of the closed points, thereby giving us an algebro-geometric alternative to the statement of part (iii) of [LQ, Theorem 5.4], and without the coprime assumption.

Parts (i) and (ii) of [LQ, Theorem 5.4] say that, whenever the rank and degree of the objects are coprime, the moduli stack of polynomial semistable objects of Chern character ch on a K3 surface is always isomorphic to a moduli scheme of stable sheaves. Note that, parts (i) and (ii) of [LQ, Theorem 5.4] can be read as statements on isomorphic moduli stacks, for if each fibre of a family of complexes is isomorphic to a sheaf at degree 0, then the family itself is isomorphic to a flat family of sheaves at degree 0; also, derived dual and shift are both functors that take families of complexes to families of complexes. Together with Theorem 3.1, we now know that under the coprime assumption on rank and degree, the moduli stack of polynomial semistable objects must be isomorphic to either a moduli scheme of stable sheaves, or a moduli stack of μ -semistable locally free sheaves.

Also, the proof of Theorem 3.1 in fact shows, that every $\mu_{\hat{\omega}}$ -semistable torsion-free sheaf on \hat{X} of Chern character \hat{ch} is the image of a σ -semistable object on X under Ψ . In other words, there are no non-locally free $\mu_{\hat{\omega}}$ -semistable sheaves of Chern character \hat{ch} on \hat{X} . Since the moduli stack of $\mu_{\hat{\omega}}$ -semistable sheaves is universally closed, we conclude that the moduli stack $\mathcal{M}_{X, ch}^{\sigma ss}$ in Theorem 3.1 is also universally closed.

Lastly, we point out that Theorem 3.1 can be considered as an extension of a theorem of Bruzzo-Maciocia: in [BM, Theorem 0.1], for a reflexive K3 surface X , they constructed an isomorphism from the Hilbert scheme $\text{Hilb}^n(X)$ of n points on X to the moduli space of Gieseker stable locally free sheaves on the Fourier-Mukai partner \hat{X} when $n \geq 1$. Choosing $\beta\omega = 0$ and using ideal sheaves, we can consider $\text{Hilb}^n(X)$ as a substack of the moduli stack of Bridgeland semistable objects of Chern character $(-1, 0, n)$. This way, our Theorem 3.1 extends Bruzzo-Maciocia's isomorphism by adding μ -semistable locally free sheaves on the \hat{X} side (also see Remark 2.4).

REFERENCES

- [Alp] J. Alper, *Good moduli spaces for Artin stacks*. arXiv:0804.2242v3 [math.AG]
- [BBR] C. Bartocci, U. Bruzzo, D. Hernández-Ruipérez, *Fourier-Mukai and Nahm Transforms in Geometry and Mathematical Physics*, Progress in Mathematics, Vol. 276, Birkhäuser, 2009.
- [Bay] A. Bayer, *Polynomial Bridgeland stability conditions and the large volume limit*, Geom. Topol., Vol. 13, pp. 2389-2425, 2009.
- [BH] M. Bernardara, G. Hein, *The Euclid-Fourier-Mukai algorithm for elliptic surfaces*, preprint, 2010. arXiv:1002.4986v2 [math.AG]
- [Bri] T. Bridgeland, *Fourier-Mukai transforms for surfaces and moduli spaces of stable sheaves*, PhD thesis, University of Edinburgh, 1998.
- [BM] U. Bruzzo and A. Maciocia, *Hilbert schemes of points on some K3 surfaces and Gieseker stable bundles*, Math. Proc. Camb. Phil. Soc., Vol. 120, pp. 255-261, 1996.

- [HP] G. Hein and D. Ploog, *Postnikov-stability versus semistability of sheaves*, preprint, 2009. arXiv:0901.1554v1 [math.AG]
- [Huy] D. Huybrechts, *Derived and Abelian equivalences of K3 surfaces*, preprint, 2006. arXiv:math/0604150v1 [math.AG]
- [HL] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics, Vol. 31, Vieweg, Braunschweig, 1997.
- [Lie] M. Lieblich, *Moduli of complexes on a proper morphism*, J. Algebraic Geom., Vol. 15, pp. 175-206, 2006.
- [LQ] J. Lo and Z. Qin, *Mini-walls for Bridgeland stability conditions on the derived category of sheaves over surfaces*, 2011, preprint. arXiv:1103.4352v1 [math.AG]
- [MM] A. Maciocia and C. Meachan, *Rank one Bridgeland stable moduli spaces on a principally polarised Abelian surface*, preprint, 2011. arXiv:1107.5304v1 [math.AG]
- [MY] H. Minamide, S. Yanagida and K. Yoshioka, *Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions*, preprint, 2011. arXiv:1106.5217v1 [math.AG]
- [MY2] H. Minamide, S. Yanagida and K. Yoshioka, *Some moduli spaces of Bridgeland's stability conditions*, preprint, 2011. arXiv:1111.6187v3 [math.AG]
- [Tod] Y. Toda, *Moduli stacks and invariants of semistable objects on K3 surfaces*, Adv. Math., Vol. 217, pp. 2736-2781, 2008.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
E-mail address: locc@missouri.edu